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INCOMPRESSIBLE BOUNDARY LAYERS WITH CONSTANT WALL SHEAR AND ARBITRARY INITIAL SHEAR PROFILES

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ABSTRACT

The problem of determining the pressure gradient required to maintain a specified wall shear in a boundary layer for a given initial shear profile is treated by solving the Crocco equation. It is found that analytic solutions are readily obtainable when the problem is restricted to the case of continuous wall shear. To illustrate the method the case of constant wall shear has been treated for initial profiles obtained from suction and injection over a flat plate, assuming that the suction or injection is terminated at the station of the initial profile. Both the pressure gradient and the downstream shear profile has been computed in each case.

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NOMENCLATURE

L	length parameter
U_∞	free stream velocity at outer edge of initial profile
U_e	free stream velocity at station X
U	velocity parallel to surface
V	velocity normal to surface
P	dynamic pressure
ρ	density
ν	kinematic viscosity
R	Reynolds No. $U_\infty L/\nu$
x	dimensionless coordinate along surface $x = X/L$
y	dimensionless coordinate normal to surface $y = (Y/L)\sqrt{R}$
u	U/U_∞
v	$(V/U_\infty)\sqrt{R}$
p	dimensionless dynamic pressure $p = P/\rho U_\infty^2$
φ	shear $\partial u/\partial y$
ξ	$x^{1/3}$
β	u/ξ
τ	wall shear $\varphi(x,0)$
ζ	$\beta/(9\tau^2)^{1/3}$

INTRODUCTION

In view of the influence of the boundary layer on the performance characteristics of flight vehicles, any method of controlling this flow region is of great interest. Skin friction, heat transfer and the location of the separation point are properties of the boundary layer which play a direct role in performance. Of these, the first two are reduced by a positive pressure gradient or by the injection of fluid normal to the wall, while separation is delayed by a negative pressure gradient and wall suction. Optimization of these properties therefore implies a compromise. In particular it may be desirable to reduce skin friction and heat transfer by injection over the frontal area of body where pressure gradient is often predetermined by other requirements. After the termination of injection or suction these properties may be kept at a suitably low level by the application of an appropriate pressure gradient. At the same time the pressure gradient must not be such as to unduly hasten separation so that a knowledge of the precise effect of the pressure gradient is needed. We are thus faced with the problem of being given an initial boundary layer flow and subsequently determining the pressure gradient required to maintain a specified wall shear from then on.

One of the earlier attempts to continue a boundarylayer determination from given initial conditions was that of Goldstein[1] who assumed an initial velocity profile together with a free stream pressure gradient and calculated the wall shear. Unfortunately the convergence properties of his solution were poor. This may have been caused by two factors in his analysis: a) a discontinuity was permitted in the wall shear at the initial station by leaving initial shear and pressure gradient completely arbitrary. In fact, the study of the behavior of the boundary layer in the neighborhood of such discontinuities was one of the purposes of the analysis. b) The initial velocity profile was specified as a power series in y, a variable with a range $0 \leq y < \infty$. It was thus impossible to represent the entire profile by a small number of terms valid over the entire range of y. While the approach of Goldstein is maintained, the above difficulties are avoided in the present paper. The first is bypassed by allowing discontinuities in the shear derivatives, but not in the shear itself. This still covers a wide variety of applications, among them the determination of downstream profiles after injection or suction is suddenly terminated. The second difficulty is circumvented by working with the Crocco equation which employs the velocity as an independent variable. Since the range of the dimensionless velocity is finite, $0 \leq u \leq 1$ it becomes feasible to represent a wide variety of initial profiles by polynomials in u valid over the entire profile. The author wishes to thank Dr. Antonio Ferri for suggesting this problem.

BASIC EQUATIONS

For two dimensional, incompressible, viscous flow over a plane surface the Prandtl boundary layer eqs. hold:

cont.
$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \qquad (1)$$

x momentum
$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = - \frac{\partial P}{\partial X} + \nu\frac{\partial^2 U}{\partial Y^2} \qquad (2)$$

y momentum
$$\frac{\partial P}{\partial Y} = 0 \qquad (3)$$

As pointed out in the introduction, however, the physical or X, Y plane is inconvenient for our purposes because of the difficulty of representing an initial condition in these variables. Furthermore, since it is eventually desired to set boundary conditions in terms of the velocity derivative at the wall $(U_Y)_{Y=0}$, it seems appropriate to work with the shear as the dependent variable[3]. It appears therefore that the Crocco transformation[2] would be suitable for the purpose at hand. The velocity is first non-dimensionalized with respect to U_1, the free stream velocity, at an initial station and a reference length L is introduced which may be characteristic of the surface geometry or perhaps a measure of the distance of the initial station from a leading edge. We introduce the dimensionless shear variable

$$\varphi(x,\ u) = \frac{\partial u}{\partial y} \tag{4}$$

together with the transformation

$$\begin{aligned} u &= u(x',\ y') \\ x &= x' \end{aligned} \tag{5}$$

of which the inverse is:

$$y' = \int_0^u \frac{du}{\varphi} \tag{5a}$$

primes referring to the physical plane. Upon elimination of v with the aid of the continuity equation the resulting transformed momentum equation for the dimensionless shear φ is the well known Crocco eq.[3]

$$\varphi^2 \varphi_{uu} = u\varphi_x - p_x \varphi_u \tag{6}$$

Like the heat conduction equation, this equation is parabolic in character, which means that conditions specified at a given station, x, have no upstream influence. It is, therefore, sufficient for the determination of a solution to set an initial condition in the form of an initial shear profile $\varphi(0, u)$ at a station which shall serve as the origin $x = 0$, together with boundary conditions.

BOUNDARY CONDITIONS

The boundary conditions to be set for the problem treated here differ somewhat from those commonly used in boundary layer theory. The shear at the wall, which is generally the quantity to be determined, is here specified beforehand:

$$\varphi(x, 0) = T(x) \tag{7}$$

the object being to determine the free stream conditions required to obtain the given wall shear. We see, therefore, that the pressure gradient characterising the free stream does not appear as a known function in eq. (6). To solve the Crocco equation, p_x will be written in a convenient functional form containing undetermined constants. It is to be understood that the general solution for φ will be in terms of these constants. Once such a solution is obtained, the condition that $v = 0$ at the wall requires that p_x satisfy the relation

$$p_x = \left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0} = (\varphi\varphi_u)_{u=0} \tag{8}$$

which serves to determine the constants appearing in p_x.

The boundary condition at the junction of the free stream with the viscous layer requires some special consideration. It has been customary to state specifically that $\varphi = 0$ when $u = u_e$ the velocity at the outer edge of the layer. Consider, however, the simpler but analogous problem of heat conduction in a semi-infinite rod. When the initial temperature approaches a finite value as $y \to \infty$, it is only necessary to specify the boundary value of the temperature at $y = 0$. No specification is imposed at $y \to \infty$. Instead the solution is required to be bounded and regular at infinity. The resultant solution for the temperature will then approach the initial temperature as $y \to \infty$. This is an expression of the physical situation that conditions at $y = 0$ cannot in a finite time influence the temperature of the rod infinitely far away by conduction. Like heat conduction the diffusion of vorticity from the wall is a parabolic phenomenon. Therefore, if the shearing stress in an initial profile approaches zero as y becomes infinite, we may suppose that in the absence of discontinuities at the outer edge of the viscous layer the shear will automatically approach zero in subsequent downstream profiles. This means that aside from an appropriate restriction on the boundedness or regularity of φ as $y \to \infty$ for $x \geq 0$ we are not required to set an explicit boundary condition at the outer edge of the boundary layer. In this context the requirement that a solution be free of singularities as $y \to \infty$ is the mathematical counterpart of the physical requirement that there be no external disturbances in the free stream just at the outer edge of the boundary layer.

To insure the proper behavior of the shear profile, i.e. $\lim_{u \to u_e} \varphi(x, u) = 0$, the initial profile cannot be completely arbitrary. To begin with as we have already implied, we must require $\varphi(0, 1) = 0$. Sufficient conditions to obtain proper solutions have been investigated by Reihnboldt[4] and discussed by others[5]. Unfortunately a formulation of the necessary conditions is not yet available and our principal guide to the correctness of a given initial profile must still be the plausibility of the results obtained. It is proposed in this paper that $\varphi(0, u)$ be represented by a power series in u

$$\varphi(0, u) = \sum_{o}^{\infty} C_n u^n \qquad (9)$$

the coefficients C_n being subject to the conditions

a) $\varphi(0, 1) = 0$

b) The variable $y = \int_0^u (\varphi/u)du$ is single valued.

c) $y \to \infty$ as $u \to 1$

For most practical applications the series is terminated and $\varphi(0, u)$ is represented by a polynomial in u for $0 \leq u < 1$. Finally the wall shear is given as a power series in $x^{1/3}$, i.e.:

$$\varphi(x, 0) = \sum_0^\infty f_n x^{n/3} \tag{10}$$

The reason for this particular choice in the form of the wall shear will become evident when the solution for φ is considered.

SOLUTION OF THE CROCCO EQUATION

Although our solution is required to be continuous at the outer edge of the boundary layer to be acceptable on physical grounds no such stipulation is made at the wall. Since initial and boundary conditions are arbitrarily specified along two intersecting lines discontinuities may occur in φ and its derivatives at the point of intersection, which in this case is taken as the origin. Employing a technique used by Goldstein[(1), (6)] specifically to investigate boundary layers with shear discontinuities at the wall, we introduce new coordinates:

$$\xi = x^{1/3} \qquad\qquad \beta = u/\xi \tag{11}$$

in terms of which the Crocco eq. (6) becomes:

$$3\varphi^2\varphi_{\beta\beta} = \beta\xi\varphi_\xi - \beta^2\varphi_\beta - 3\xi p_x\varphi_\beta \tag{12}$$

The pressure gradient p_x will be represented in the general form:

$$p_x = \sum_{0}^{\infty} A_n \xi^n \tag{13}$$

In compliance with earlier remarks regarding the nature of φ at the outer edge of the layer we seek solutions to eq. (12) in the form of a series

$$\varphi(\xi, \beta) = \sum_{0}^{\infty} \varphi_n(\beta) \xi^n \tag{14}$$

After substitution of eq. (14) into (11) and equating the coefficients of like powers of ξ, equations are obtained for the φ_n.

$$\begin{aligned}
&3\varphi_o^2\varphi_o''(\beta) + \beta^2\varphi_o'(\beta) = 0 \\
&3\varphi_o^2\varphi_1'' + \beta^2\varphi_1' + (6\varphi_o\varphi_o'' - \beta)\varphi_1 = -3A_o\varphi_o' \\
&\quad\vdots \\
&3\varphi_o^2\varphi_n'' + \beta^2\varphi_n' + (6\varphi_o\varphi_o'' - n\beta)\varphi_n = -3A_o\varphi_{n-1}' \\
&- 3\sum_{k=1}^{n-1}\left\{A_k\varphi_{n-1-k}' + \varphi_o''\varphi_k\varphi_{n-1-k} + \varphi_k''\sum_{j=0}^{n-k}\varphi_j\varphi_{n-j-k}\right\}
\end{aligned} \tag{15}$$

To translate the boundary and initial conditions into boundary conditions for φ_n we note first that in the x, u plane lines of β = const. are a family of curves passing through the origin. The wall corresponding to $\beta = 0$ while the u axis is approached as $\beta \to \infty$, $\xi \to 0$ with u finite. Setting $\beta = 0$ in eq. (14) and comparing it with the boundary condition eq. (10) we immediately obtain

$$\varphi_n(0) = f_n \tag{16}$$

The second condition is found by writing eq. (14) in the form

$$\varphi(\xi, \beta) = \sum_{0}^{\infty} \frac{\varphi_n(\beta)}{\beta^n} u^n \tag{17}$$

which after letting β tend to infinity and comparing with eq. 9 yields

$$\lim_{\beta\to\infty} \left[\frac{\varphi_n(\beta)}{\beta^n}\right] = C_n \tag{18}$$

Equations (15) begin with a non-linear equation for φ_o followed by inhomogeneous linear equations for φ_n in which φ_o and its derivatives appear as coefficients. Thus, unless φ_o is simple in form it would be necessary to resort to numerical methods to solve the system. From the boundary condition eqs. (16) and (18) we see that $\varphi_o(0) = f_o$ and $\lim_{\beta\to\infty} \varphi_o = C_o$. If we limit ourselves to cases for which the wall shear is continuous at the origin i.e.: for which $f_o = C_o = \tau$, it is evident that a solution for φ_o satisfying initial and boundary conditions is:

$$\varphi_o = \tau \tag{19}$$

In this event the differential equation for φ_1 becomes:

$$3\tau^2\varphi_1'' + \beta^2\varphi_1' - \beta\varphi_1 = 0 \tag{20}$$

By inspection it can immediately be seen that one solution to eq. (20) is $\varphi_1 = \beta$ and that a complete solution is obtained after application of eqs. (16) and (18)

$$\varphi_1 = C_1\beta + f_1\beta \int_{\beta}^{\infty} \frac{e^{-\beta^3/9\tau^2}}{\beta^2}\, d\beta \tag{21}$$

In the analysis that follows we shall limit ourselves to the case of constant wall shear, i.e.:

$$f_o = \tau$$
$$f_n = 0 \qquad n \geq 1 \tag{22}$$

for which we have

$$\varphi_1 = C_1 \beta \tag{23}$$

The equations for subsequent function φ_n may then be written

$$3\tau^2 \varphi_n'' + \beta^2 \varphi_n - n\beta\varphi_n = r_n(\beta) \tag{24}$$

where

$$r_n(\beta) = -3A_o \varphi'_{n-1} - 3\sum_{k=1}^{n-1} \left\{A_k \varphi'_{n-1-k} + \varphi''_k \sum_{j=0}^{n-k} \varphi_j \varphi_{n-k-j}\right\} \tag{25}$$

In view of the boundary condition expressed by eq. (18) we introduce the new variable $\psi_n = \varphi_n/\beta^n$ obtaining from eq. (24)

$$3\tau^2 \psi_n'' + \left(\beta^2 + \frac{6\tau^2 n}{\beta}\right)\psi_n' + \frac{3n(n-1)\tau^2}{\beta^2}\,\psi = r_n(\beta) \tag{26}$$

for which two homogeneous solutions may be found[7]

$$\psi_n^{(1)} = \beta^{1-n}\,{}_1F_1\left\{\frac{1-n}{3}, \frac{4}{3}, -\frac{\beta^3}{9\tau^2}\right\}$$
$$\psi_n^{(2)} = \beta^{-n}\,{}_1F_1\left\{-\frac{n}{3}, \frac{2}{3}, -\frac{\beta^3}{9\tau^2}\right\} \tag{27}$$

${}_1F_1\left\{a, b, z\right\}$ representing the confluent hypergeometric function. Knowing the homogeneous solution the complete solution to eqs. (26) or (25) may be written for $n \geq 2$

$$\varphi_n(\beta) = D_n H_n(\zeta) + E_n Y_n(\zeta) + \frac{(9\tau^2)^{2/3}}{3\tau^2} Y_n(\zeta) \int_0^{\zeta} \frac{e^{-\zeta_1^3}}{Y_n^2(\zeta_1)} \left(\int_0^{\zeta_1} e^{\zeta_2^3} r_n(\zeta_2) Y_n(\zeta_2) d\zeta_2 \right) d\zeta_1 \tag{28}$$

where

$$\zeta = \beta/(9\tau^2)^{1/3}$$

$$H_n(\zeta) = \zeta_1 F_1\left\{\frac{1-n}{3}, \frac{4}{3}, -\zeta^3\right\}$$

$$Y_n(\zeta) = {}_1F_1\left\{-\frac{n}{3}, \frac{2}{3}, -\zeta^3\right\}$$

Application of the boundary condition at the wall eq. (16) yields in view of the restriction eqs. (22) representing the constancy of the wall shear

$$E_n = 0 \tag{29}$$

The requirement of $v = 0$ at the wall, eq. (8) together with eqs. (13) and (29) determine the coefficients A_n appearing in the pressure gradient. Thus,

$$\sum_0^{\infty} A_n \xi^n = \tau \sum_1^{\infty} \varphi_n^1(0) \xi^{n-1} = \frac{\tau}{(9\tau^2)^{1/3}} \sum_1^{\infty} D_n \xi^{n-1} \tag{30}$$

or

$$A_n = \frac{\tau}{(9\tau^2)^{1/3}} D_{n+1} \tag{31}$$

It now remains to determine D_n by substitution of eq. (28) with $E_n = 0$ into the initial condition eq. (18) and solving for D_n.

$$D_n = (9\tau^2)^{\frac{n}{3}} \cdot \frac{\Gamma(1+\frac{n}{3})}{\Gamma(4/3)} \left[C_n - \frac{\Gamma(2/3)}{3\tau^2(9\tau^2)^{\frac{n}{3}}\Gamma(\frac{2+n}{3})} \int_0^\infty \frac{e^{-\zeta_1^3}}{Y_n^2(\zeta_1)} \left[\int_0^{\zeta_1} r_n(\zeta_2) Y_n(\zeta_2) e^{\zeta_2^3} d\zeta_2 \right] d\zeta_1 \right] \tag{32}$$

where we note that since r_n contains the coefficients $D_o, \cdots D_{n-1}$ ($D_o = (9\tau^2)^{1/3}\tau$, with $D_1 = (9\tau^2)^{1/3}C\tau$) the constants D_n may be evaluated successively. The complete solution for $\varphi_2(\beta)$ in particular is more readily arrived at by direct consideration of eq. (15) for n = 2.

$$3\tau^2\varphi_2'' + \beta^2\varphi_2' - 2\beta\varphi_2 = - \frac{3\tau D_1^2}{(9\tau^2)^{2/3}} \tag{33}$$

By inspection a particular solution is seen to be $- D_1^2\beta^2/2\tau(9\tau^2)^{2/3}$ so that the complete solution after setting $E_2 = 0$ is:

$$\varphi_2(\zeta) = D_2 H_2(\zeta) - \frac{D_1^2\zeta^2}{2\tau} \tag{34}$$

D_2 being determined from the boundary condition eq. (32)

$$D_2 = (9\tau^2)^{2/3} \frac{\Gamma(5/3)}{\Gamma(4/3)} \left[C_2 - \frac{D_1^2}{2\tau} \right] \tag{35}$$

Evaluating $r_3(\zeta)$ and employing recursion relations for the functions ${}_1F_1$ we have

$$r_3(\zeta) = - \frac{3\tau D_1 D_2}{(9\tau^2)^{2/3}} \cdot F_{3,1}(\zeta) + \frac{9D_1^3}{(9\tau^2)^{2/3}} \cdot F_{3,2}(\zeta) \tag{36}$$

$$F_{3,1}(\zeta) = 1 + (1+6\zeta^3)\,{}_1F_1\left\{-\frac{1}{3}, \frac{4}{3}, -\zeta^3\right\} + \frac{3}{4}(1-6\zeta^3)\zeta^3\,{}_1F_1\left\{\frac{2}{3}, \frac{7}{3}, -\zeta^3\right\}$$

$$F_{3,2}(\zeta) = \zeta$$

$\varphi_3(\zeta)$ is then obtained from eq. (28) with n = 3

$$\varphi_3(\zeta) = D_3 H_3(\zeta) - Y_3(\zeta)\left[\frac{D_1 D_2}{T} G_{3,1}(\zeta) - \frac{3D_1^3}{T^2} G_{3,2}(\zeta)\right] \tag{37}$$

where in general

$$G_{i,j}(\zeta) = \int_0^\zeta \frac{e^{-\zeta_1^3} \int_0^{\zeta_1} Y_i(\zeta_2) F_{i,j}(\zeta_2) e^{\zeta_2^3} d\zeta_2}{Y_i^2(\zeta_1)} d\zeta_1 \tag{38}$$

similarly $\varphi_4(\zeta)$ may be evaluated

$$\varphi_4(\zeta) = D_4 H_4(\zeta) - Y_4(\zeta)\left\{\frac{D_3 D_1}{T} G_{4,1}(\zeta) + \frac{D_2^2}{T} G_{4,2}(\zeta)\right.$$

$$\left. - \frac{D_1^2 D_2}{T^2} G_{4,3}(\zeta) + \frac{3D_1^4}{T^3} G_{4,4}(\zeta)\right\} \tag{39}$$

for which

$$F_{4,1}(\zeta) = 1 + H_3'(\zeta) + 2\zeta H_3''(\zeta)$$

$$F_{4,2}(\zeta) = H_2(\zeta) + 2H_2(\zeta) H_2''(\zeta)$$

$$F_{4,3}(\zeta) = \zeta + 2H_2(\zeta) + (Y_3 G_{3,1})' + 2\zeta(Y_3 G_{3,1})''$$

$$F_{4,4}(\zeta) = Y_3 G_{3,2})' + 2\zeta(Y_3 G_{3,2})''$$

Carrying out the differentiations indicated for $F_{4,3}$ and $F_{4,4}$ we have in general

$$(Y_i G_{i,j})' = \frac{e^{-\zeta^3} \int_0^\zeta Y_i(\zeta_1) F_{i,j}(\zeta_1) e^{\zeta_1^3} d\zeta_1}{Y_i(\zeta)} + Y_i'(\zeta) G_{i,j}(\zeta)$$

$$(Y_i G_{i,j})'' = Y_i''(\zeta) G_{i,j}(\zeta) - \frac{3\zeta^2 e^{-\zeta^3}}{Y_i(\zeta)} \int_0^\zeta Y_i(\zeta_1) F_{i,j}(\zeta_1) e^{\zeta_1^3} d\zeta_1 + F_{i,j}(\zeta) \tag{40}$$

Finally the coefficients D_3 and D_4 are evaluated from eq. (32)

$$D_3 = \frac{9T^2}{\Gamma(4/3)}\left[C_3 + \frac{3}{2}\cdot\frac{1}{(9T^2)}\left(\frac{D_1 D_2}{T}G_{3,1}(\infty) - \frac{3D_1^3}{T^2}G_{3,2}(\infty)\right)\right]$$

$$D_4 = \frac{(9T^2)^{4/3}\Gamma(7/3)}{\Gamma(4/3)}\left[C_4 - \frac{\Gamma(2/3)}{(9T^2)^{4/3}}\left(\frac{D_3 D_1}{T}G_{4,1}(\infty) + \frac{D_2^2}{T}G_{4,2}(\infty) - \frac{D_1^2 D_2}{T^2}G_{4,3}(\infty) + \frac{3D_1^4}{T^3}G_{4,4}(\infty)\right)\right] \tag{41}$$

Although the expressions for $n > 4$ are increasingly complex,no qualitatively new features are introduced and higher order terms may be evaluated in the same manner as φ_4.

EVALUATION OF THE FUNCTIONS $G_{i,j}(\zeta)$

The formulation presented above has the advantage of yielding solutions for $\varphi_n(\beta)$ in closed form. Unfortunately the integrals appearing in the form of $G_{i,j}(\zeta)$ are generally too complex to be evaluated analytically. However, recursion relations materially reduce the number of functions $H_n(\zeta)$ to be computed and a careful consideration of the specific form of the double integrals involved indicates that the numerical evaluation of $G_{i,j}(\zeta)$ is quite feasible.

An immediate difficulty encountered if we simply attempt to evaluate the first integral of $G_{i,j}$ is the magnitude of the exponential term. Since $G_{i,j}$ approaches its asymptotic value closely only for $\zeta > 5$ it is evident that we would be forced to compute numbers of the magnitude of exp. (100) and larger, making evaluation by computers impractical. If on the other hand we deal directly with the function

$$EI_{i,j}(\zeta) = e^{-\zeta^3}\int_0^\zeta e^{\zeta_1^3}Y_i(\zeta_1)F_{i,j}(\zeta_1)d\zeta_1 \tag{42}$$

this difficulty may be circumvented. Breaking the field of integration into N small fields of extent $\Delta\zeta$ and writing

$$EI_{i,j}(N) = e^{-(N\Delta\zeta)^3}\int_0^{N\Delta\zeta} e^{\zeta_1^3} Y_i(\zeta_1)F_{i,j}(\zeta_1)d\zeta_1 \tag{43}$$

we note that this may be further broken down.

$$EI_{i,j}(N) = e^{-(N\Delta\zeta)^3}\int_{(N-1)\Delta\zeta}^{N\Delta\zeta} e^{\zeta_1^3} Y_i(\zeta_1)F_{i,j}(\zeta_1)d\zeta_1 + e^{-\left\{[N\Delta\zeta]^3-[(N-1)\Delta\zeta]^3\right\}}EI_{i,j}(N-1) \tag{44}$$

If $\Delta\zeta$ is sufficiently small, the first integral on the right side of eq. (44) may be approximated to obtain

$$EI_{i,j}(n) \approx e^{-\left\{[N\Delta\zeta]^3-[(N-\frac{1}{2})\Delta]^3\right\}} Y_1[(N-\tfrac{1}{2})\Delta\zeta]F_{i,j}[N-\tfrac{1}{2})\Delta\zeta]\Delta\zeta +$$

$$+ e^{-\left\{[N\Delta\zeta]^3-[(N-1)\Delta\zeta]^3\right\}}EI_{i,j}(N-1) \tag{45}$$

with $EI_{i,j}(1) = e^{-\frac{7}{8}\Delta\zeta^3} Y_1(\frac{\Delta\zeta}{2})F_{i,j}(\frac{\Delta(\zeta)}{2})\Delta\zeta$

For any given step $\Delta\zeta$ the exponentials involved are now considerably smaller and $EI_{i,j}(\zeta)$ may be obtained through a step by step procedure. $G_{i,j}(\zeta)$ is obtained by direct numerical integration of

$$G_{i,j}(\zeta) = \int_0^{\zeta} \frac{E_{i,j}(\zeta_1)}{Y_i^2(\zeta_1)}\, d\zeta_1 \tag{46}$$

Conveniently the functions $EI_{i,j}$ evaluated in the above manner are precisely the functions appearing in $(Y_iG_{i,j})'$ and $(Y_iG_{i,j})''$ eqs. (40) which must be computed for the $G_{i+1,j}$ etc.

For this paper the $G_{i,j}$ have been computed on an IBM 1620 computer and are presented in Table 1 and Figure 3. Estimated accuracy is within two percent for the chosen interval of $\Delta\zeta = .05$.

APPLICATIONS

The continuation of the asymptotic suction profile[8] is a particularly good application of the above theory.. The initial shear profile is given exactly by the expression

$$\varphi(0,u) = 2(1-u) \tag{47}$$

The characteristic length chosen is the distance from the leading edge of the plate to the position at which the asymptotic form of the profile is to all intents and purposes reached[9]. Downstream of the profile represented by eq. (47) the uniform suction is terminated and shear profiles for x = .2 and .4 have been computed and presented in Fig. (1). The pressure gradient for constant wall shear $\varphi(x,0) = 2$ is in this case

$$p_x = -\frac{2}{(36)^{1/3}}[6.6039 - 11.014x^{1/3} + 11.579x^{2/3} - 5.771x + \ldots] \tag{48}$$

Terms up to $\varphi_3(\zeta)\xi^3$ were used in computing the shear, a third order expression in this instance giving better results than a fourth order one. The computation of φ_4 involved a small difference between two large numbers occurring in the coefficient of Y_4 thus requiring very high accuracy in the determination of $G_{i,j}(\zeta)$. This situation does not arise however in the determination of p_x for which terms up to $\tau D_4\xi^3/(9\tau^2)^{1/3}$ are used. As a check u_e was determined by integrating eq. (48) for p_x and employing the Bernoulli relation for the outer edge of the boundary layer.

$$p + \frac{u_e^2}{2} = \tfrac{1}{2} \tag{49}$$

From Fig. (1) it is seen that the value of u for which φ becomes zero agrees very well with u_e obtained from eq. (49) thus bearing out the remarks in the section dealing with boundary conditions.

A second application of importance is the continuation of a profile on a flat plate with injection[10] up to our initial station at a velocity

$$V_o = -\, C_q^* U_\infty / 2\sqrt{R_x}$$

or

$$v = -\, C_q^*/2 \tag{50}$$

For this particular case we have chosen $C_q^* = -\frac{1}{2}$ with L again chosen as the distance from the leading edge to the initial station. A cubic polynomial in u was found to provide a good fit to the initial profile as seen in Fig. (2). Again terms up to $\varphi_3 \xi^3$ were used in computing downstream profiles at x = .4, .8 and 1.2. The pressure gradient required for constant wall shear was found to be for $\varphi = .175$:

$$p_x = \frac{.175}{(9\tau^2)^{1/3}} [.0846 + .1854x^{1/3} - .09722x^{2/3} - .02677x] \tag{51}$$

It is interesting to note that the p_x is initially negative for the suction profile and positive for the injection profile. This is to be expected on physical grounds since in the suction profile we would expect in the absence of a pressure gradient that vorticity would tend to diffuse away from the wall where it is highest. To maintain the vorticity or shear we would therefore need a drop in pressure. For the injection profile the maximum vorticity is found at a point above the wall. We would, therefore, expect vorticity to tend to diffuse away from this point toward the wall (as well as toward the free stream). To keep the shear at the wall from increasing, an increase in pressure is, therefore, required initially.

CONCLUSION

A method of determining the pressure gradient required to maintain a constant wall shear in a boundary layer when given an initial shear profile has been developed. The method also yields an analytic result for downstream shear profiles in terms of tabulated universal functions. The theory has been applied to the cases of profiles derived from a) constant suction over a flat plate and b) injection normal to a flat plate with velocity v proportional to $1/\sqrt{x}$.

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TABLE 1

Z	G31	G32	G41	G42	G43	G44
0.075	0.00500	0.00006	0.00500	0.00250	0.00056	0.00000
0.125	0.01497	0.00031	0.01497	0.00748	0.00280	0.00005
0.175	0.02984	0.00087	0.02984	0.01492	0.00782	0.00018
0.225	0.04944	0.00184	0.04943	0.02472	0.01661	0.00050
0.275	0.07348	0.00334	0.07348	0.03674	0.03006	0.00112
0.325	0.10156	0.00541	0.10156	0.05078	0.04885	0.00214
0.375	0.13311	0.00811	0.13317	0.06658	0.07339	0.00368
0.425	0.16749	0.01144	0.16767	0.08383	0.10381	0.00583
0.475	0.20395	0.01536	0.20440	0.10219	0.13993	0.00864
0.525	0.24173	0.01978	0.24269	0.12133	0.18131	0.01215
0.575	0.28009	0.02462	0.28190	0.14092	0.22730	0.01631
0.625	0.31835	0.02975	0.32147	0.16068	0.27713	0.02107
0.675	0.35595	0.03504	0.36095	0.18039	0.32995	0.02634
0.725	0.39243	0.04037	0.39996	0.19984	0.38492	0.03200
0.775	0.42744	0.04563	0.43822	0.21889	0.44125	0.03794
0.825	0.46077	0.05074	0.47553	0.23744	0.49821	0.04405
0.875	0.49227	0.05563	0.51171	0.25537	0.55513	0.05022
0.925	0.52187	0.06023	0.54666	0.27264	0.61144	0.05636
0.975	0.54954	0.06454	0.58026	0.28917	0.66663	0.06240
1.025	0.57530	0.06853	0.61244	0.30492	0.72026	0.06828
1.075	0.59919	0.07220	0.64314	0.31985	0.77202	0.07396
1.125	0.62127	0.07555	0.67232	0.33392	0.82157	0.07940
1.175	0.64158	0.07861	0.69993	0.34713	0.86871	0.08458
1.225	0.66022	0.08139	0.72596	0.35945	0.91330	0.08949
1.275	0.67726	0.08391	0.75041	0.37089	0.95525	0.09412
1.325	0.69279	0.08619	0.77330	0.38147	0.99454	0.09847
1.375	0.70691	0.08825	0.79465	0.39120	1.03121	0.10256
1.425	0.71970	0.09011	0.81452	0.40013	1.06532	0.10638
1.475	0.73128	0.09179	0.83295	0.40829	1.09697	0.10994
1.525	0.74173	0.09331	0.85002	0.41572	1.12630	0.11327
1.575	0.75116	0.09468	0.86581	0.42249	1.15344	0.11636
1.625	0.75967	0.09592	0.88037	0.42864	1.17854	0.11924
1.675	0.76734	0.09705	0.89380	0.43422	1.20174	0.12192
1.725	0.77426	0.09807	0.90618	0.43930	1.22319	0.12440
1.775	0.78051	0.09899	0.91758	0.44390	1.24303	0.12671
1.825	0.78615	0.09983	0.92807	0.44809	1.26137	0.12885
1.875	0.79126	0.10060	0.93774	0.45189	1.27835	0.13084
1.925	0.79589	0.10130	0.94664	0.45536	1.29407	0.13269
1.975	0.80010	0.10193	0.95484	0.45852	1.30863	0.13441
2.025	0.80392	0.10251	0.96240	0.46140	1.32212	0.13601
2.075	0.80707	0.10304	0.96938	0.46369	1.33411	0.13749
2.125	0.80978	0.10353	0.97581	0.46563	1.34514	0.13887
2.175	0.81217	0.10398	0.98175	0.46732	1.35540	0.14016
2.225	0.81433	0.10439	0.98724	0.46884	1.36497	0.14135
2.275	0.81629	0.10476	0.99232	0.47022	1.37391	0.14247
2.325	0.81808	0.10511	0.99701	0.47149	1.38223	0.14351
2.375	0.81973	0.10543	1.00136	0.47265	1.38998	0.14448
2.425	0.82124	0.10572	1.00539	0.47371	1.39720	0.14538
2.475	0.82263	0.10599	1.00913	0.47469	1.40393	0.14623
2.525	0.82391	0.10624	1.01261	0.47560	1.41020	0.14702

TABLE 1

Z	G31	G32	G41	G42	G43	G44
2.575	0.82509	0.10648	1.01583	0.47643	1.41605	0.14776
2.625	0.82618	0.10669	1.01883	0.47720	1.42150	0.14845
2.675	0.82719	0.10689	1.02162	0.47792	1.42660	0.14910
2.725	0.82813	0.10708	1.02421	0.47858	1.43136	0.14971
2.775	0.82900	0.10725	1.02663	0.47919	1.43581	0.15028
2.825	0.82981	0.10741	1.02889	0.47976	1.43998	0.15081
2.875	0.83056	0.10756	1.03099	0.48029	1.44388	0.15131
2.925	0.83125	0.10770	1.03295	0.48078	1.44753	0.15178
2.975	0.83190	0.10783	1.03479	0.48124	1.45096	0.15222
3.025	0.83250	0.10795	1.03650	0.48166	1.45417	0.15264
3.075	0.83307	0.10806	1.03811	0.48206	1.45718	0.15303
3.125	0.83359	0.10817	1.03961	0.48243	1.46001	0.15340
3.175	0.83408	0.10826	1.04101	0.48278	1.46267	0.15374
3.225	0.83454	0.10836	1.04233	0.48310	1.46517	0.15407
3.275	0.83497	0.10844	1.04356	0.48340	1.46751	0.15438
3.325	0.83537	0.10852	1.04472	0.48368	1.46972	0.15467
3.375	0.83574	0.10860	1.04580	0.48394	1.47180	0.15494
3.425	0.83610	0.10867	1.04682	0.48419	1.47375	0.15520
3.475	0.83642	0.10873	1.04778	0.48442	1.47559	0.15544
3.525	0.83673	0.10880	1.04867	0.48464	1.47733	0.15567
3.575	0.83702	0.10886	1.04952	0.48484	1.47896	0.15589
3.625	0.83729	0.10891	1.05031	0.48503	1.48050	0.15609
3.675	0.83755	0.10896	1.05106	0.48521	1.48196	0.15629
3.725	0.83779	0.10901	1.05176	0.48537	1.48333	0.15647
3.775	0.83801	0.10906	1.05242	0.48553	1.48462	0.15664
3.825	0.83822	0.10910	1.05304	0.48568	1.48584	0.15680
3.875	0.83842	0.10914	1.05362	0.48582	1.48699	0.15696
3.925	0.83861	0.10918	1.05418	0.48595	1.48808	0.15710
3.975	0.83879	0.10921	1.05469	0.48607	1.48910	0.15724
4.025	0.83895	0.10925	1.05518	0.48619	1.49007	0.15737
4.075	0.83911	0.10928	1.05564	0.48629	1.49099	0.15750
4.125	0.83925	0.10931	1.05608	0.48640	1.49185	0.15761
4.175	0.83939	0.10934	1.05649	0.48649	1.49267	0.15772
4.225	0.83952	0.10936	1.05687	0.48658	1.49345	0.15783
4.275	0.83964	0.10939	1.05723	0.48667	1.49418	0.15793
4.325	0.83976	0.10941	1.05758	0.48675	1.49487	0.15802
4.375	0.83987	0.10943	1.05790	0.48682	1.49552	0.15811
4.425	0.83997	0.10945	1.05821	0.48689	1.49613	0.15819
4.475	0.84007	0.10947	1.05849	0.48696	1.49672	0.15827
4.525	0.84016	0.10949	1.05876	0.48702	1.49727	0.15835
4.575	0.84025	0.10951	1.05902	0.48708	1.49779	0.15842
4.625	0.84033	0.10953	1.05926	0.48714	1.49828	0.15849
4.675	0.84040	0.10954	1.05949	0.48719	1.49875	0.15855
4.725	0.84047	0.10956	1.05970	0.48724	1.49919	0.15861
4.775	0.84054	0.10957	1.05991	0.48729	1.49961	0.15867
4.825	0.84061	0.10958	1.06010	0.48733	1.50000	0.15872
4.875	0.84067	0.10960	1.06028	0.48737	1.50037	0.15877
4.925	0.84072	0.10961	1.06045	0.48741	1.50072	0.15882
4.975	0.84078	0.10962	1.06061	0.48745	1.50106	0.15887
5.025	0.84083	0.10963	1.06076	0.48749	1.50137	0.15891

TABLE 1

Z	G31	G32	G41	G42	G43	G44
5.125	0.84092	0.10965	1.06104	0.48755	1.50195	0.15899
5.175	0.84096	0.10966	1.06116	0.48758	1.50221	0.15903
5.225	0.84100	0.10966	1.06128	0.48761	1.50247	0.15906
5.275	0.84104	0.10967	1.06139	0.48763	1.50270	0.15910
5.325	0.84108	0.10968	1.06150	0.48766	1.50293	0.15913
5.375	0.84111	0.10969	1.06160	0.48768	1.50314	0.15916
5.425	0.84114	0.10969	1.06169	0.48770	1.50334	0.15918
5.475	0.84117	0.10970	1.06178	0.48772	1.50352	0.15921
5.525	0.84120	0.10970	1.06187	0.48774	1.50370	0.15923
5.575	0.84123	0.10971	1.06194	0.48776	1.50387	0.15926
5.625	0.84125	0.10971	1.06202	0.48778	1.50403	0.15928
5.675	0.84127	0.10972	1.06209	0.48779	1.50418	0.15930
5.725	0.84130	0.10972	1.06215	0.48781	1.50432	0.15932
5.775	0.84132	0.10973	1.06222	0.48782	1.50445	0.15934
5.825	0.84134	0.10973	1.06227	0.48784	1.50458	0.15936
5.875	0.84135	0.10974	1.06233	0.48785	1.50470	0.15937
5.925	0.84137	0.10974	1.06238	0.48786	1.50481	0.15939
5.975	0.84139	0.10974	1.06243	0.48787	1.50491	0.15940
6.025	0.84140	0.10975	1.06248	0.48788	1.50501	0.15942
6.075	0.84142	0.10975	1.06252	0.48789	1.50510	0.15943
6.125	0.84143	0.10975	1.06256	0.48790	1.50519	0.15944
6.175	0.84144	0.10975	1.06260	0.48791	1.50527	0.15946
6.225	0.84146	0.10976	1.06263	0.48792	1.50535	0.15947
6.275	0.84147	0.10976	1.06266	0.48792	1.50542	0.15948
6.325	0.84148	0.10976	1.06270	0.48793	1.50549	0.15949
6.375	0.84149	0.10976	1.06273	0.48794	1.50556	0.15950
6.425	0.84150	0.10976	1.06275	0.48795	1.50562	0.15950
6.475	0.84151	0.10977	1.06278	0.48795	1.50568	0.15951
6.525	0.84151	0.10977	1.06280	0.48796	1.50573	0.15952
6.575	0.84152	0.10977	1.06283	0.48796	1.50578	0.15953
6.625	0.84153	0.10977	1.06285	0.48797	1.50583	0.15953
6.675	0.84154	0.10977	1.06287	0.48797	1.50587	0.15954
6.725	0.84154	0.10977	1.06289	0.48798	1.50591	0.15955
6.775	0.84155	0.10977	1.06290	0.48798	1.50595	0.15955
6.825	0.84155	0.10978	1.06292	0.48798	1.50599	0.15956
6.875	0.84156	0.10978	1.06294	0.48799	1.50602	0.15956
6.925	0.84156	0.10978	1.06295	0.48799	1.50606	0.15957
6.975	0.84157	0.10978	1.06296	0.48799	1.50609	0.15957
7.025	0.84157	0.10978	1.06298	0.48800	1.50612	0.15957
7.075	0.84158	0.10978	1.06299	0.48800	1.50614	0.15958
7.125	0.84158	0.10978	1.06300	0.48800	1.50617	0.15958
7.175	0.84158	0.10978	1.06301	0.48800	1.50619	0.15959
7.225	0.84159	0.10978	1.06302	0.48801	1.50621	0.15959
7.275	0.84159	0.10978	1.06303	0.48801	1.50623	0.15959
7.325	0.84159	0.10978	1.06304	0.48801	1.50625	0.15959
7.375	0.84160	0.10978	1.06305	0.48801	1.50627	0.15960
7.425	0.84160	0.10978	1.06305	0.48801	1.50629	0.15960
7.475	0.84160	0.10978	1.06306	0.48802	1.50630	0.15960
7.525	0.84160	0.10979	1.06307	0.48802	1.50632	0.15960
10.425	0.84163	0.10979	1.06315	0.48804	1.50651	0.15963

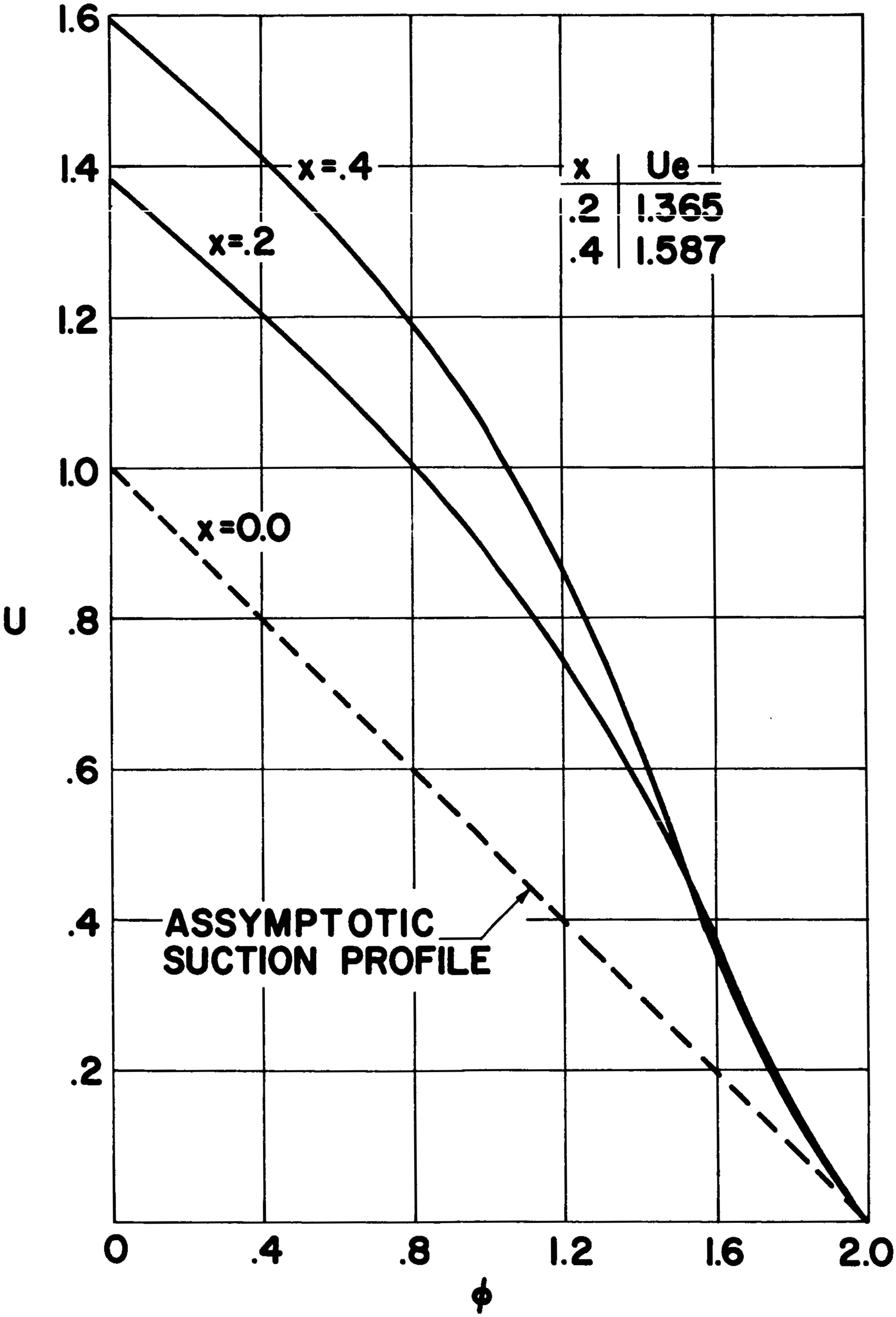

FIG. 1 SHEAR PROFILES FOR CONSTANT WALL SHEAR WITH ASYMPTOTIC SUCTION PROFILE AT x = 0

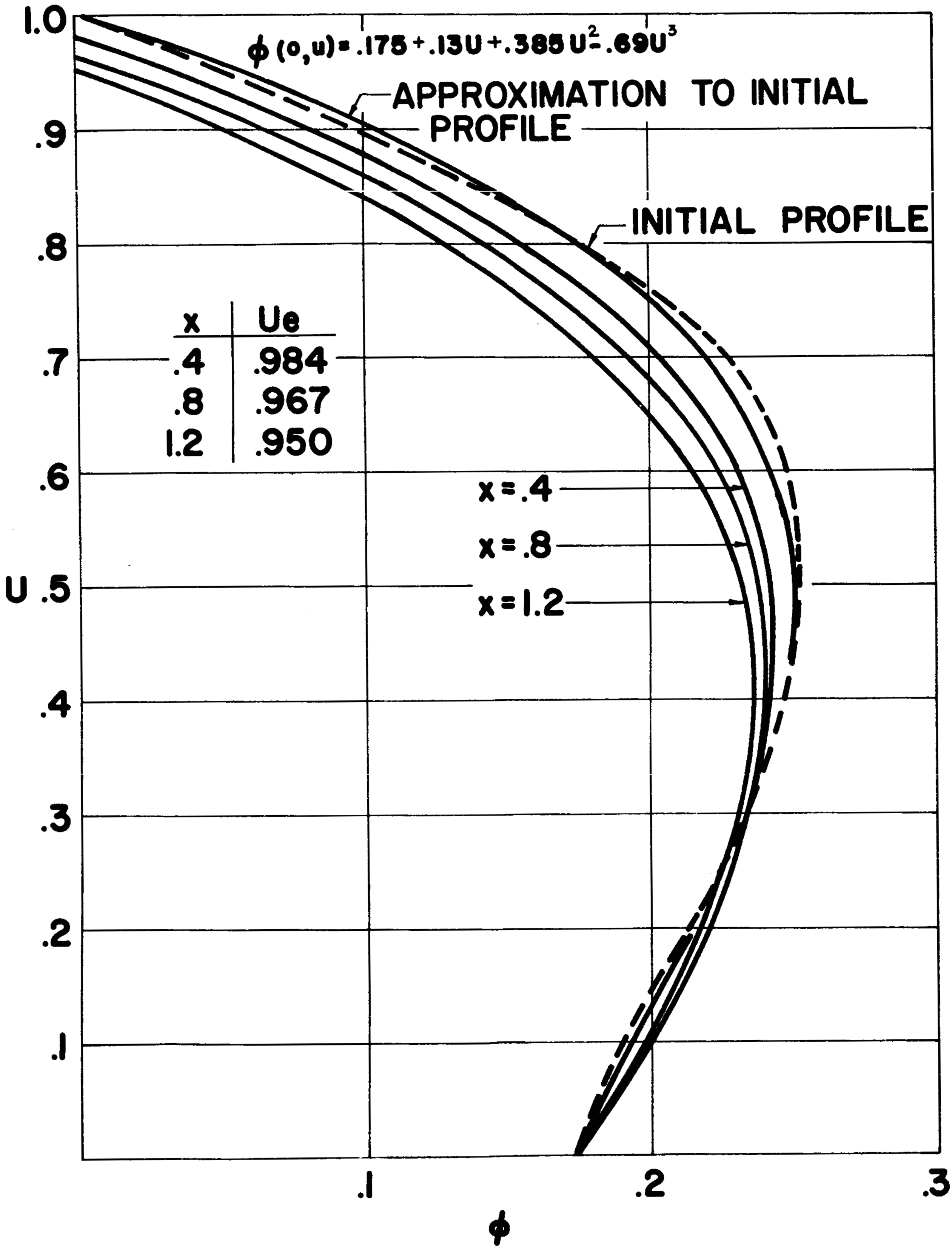

FIG. 2 SHEAR PROFILES FOR CONSTANT WALL SHEAR WITH INJECTION PROFILE AT x = 0

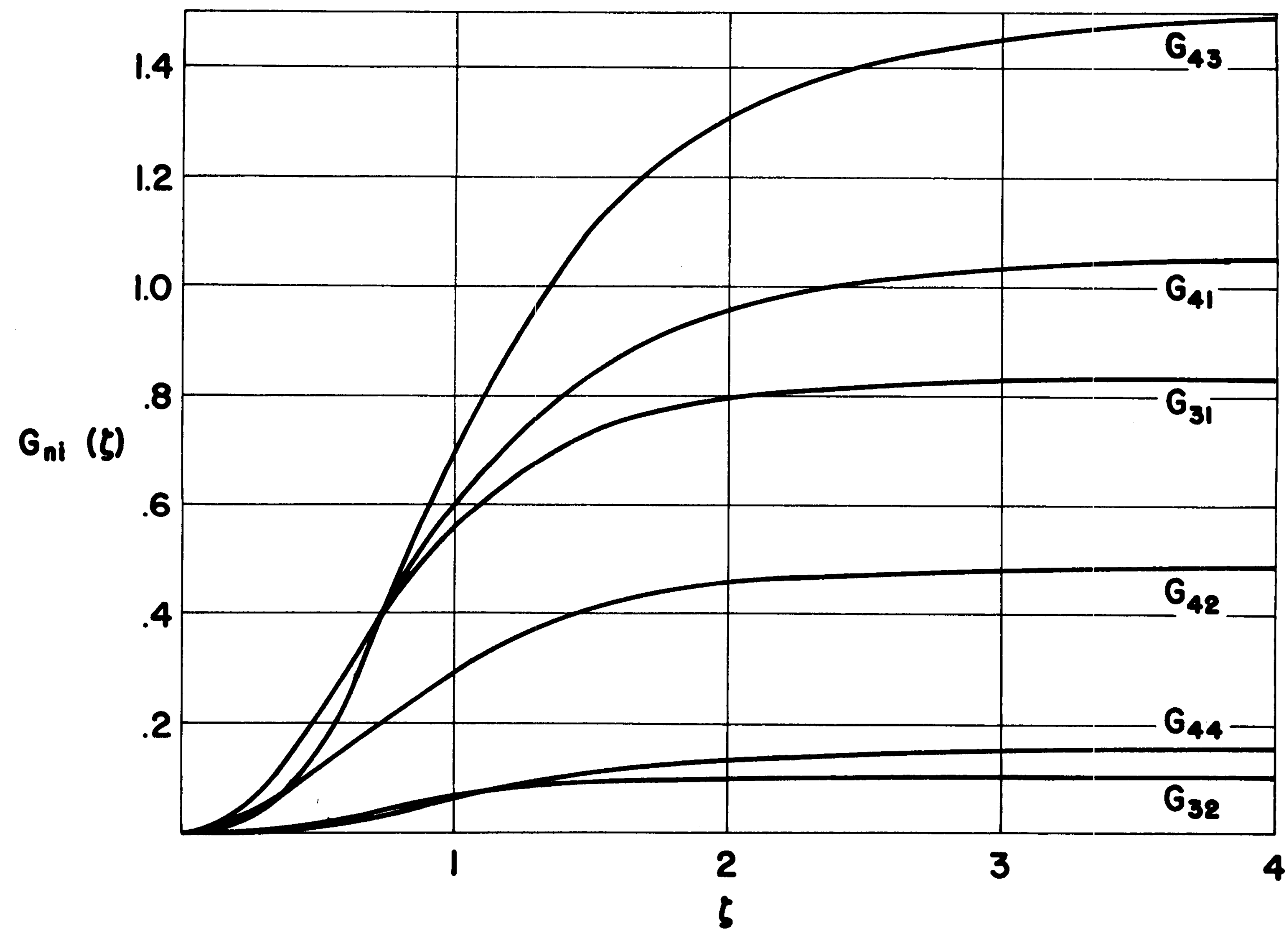

FIG. 3 $G_{i,j}(\zeta)$